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# Coexistence of quantum operations

**Teiko Heinosaari**[1], **Daniel Reitzner**[2], **Peter Stano**[2] **and Mario Ziman**[2,3]

[1] Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen, Denmark
[2] Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia
[3] Faculty of Informatics, Masaryk University, Botanická 68a, Brno, Czech Republic

E-mail: heinosaari@nbi.dk, daniel.reitzner@savba.sk, peter.stano@savba.sk and ziman@savba.sk

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**Abstract**
Quantum operations are used to describe the observed probability distributions and conditional states of the measured system. In this paper, we address the problem of their joint measurability (coexistence). We derive two equivalent coexistence criteria. The two most common classes of operations—Lüders operations and conditional state preparators—are analyzed. It is shown that Lüders operations are coexistent only under very restrictive conditions, when the associated effects are either proportional to each other or disjoint.

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## 1. Introduction

One of the basic implications of quantum mechanics is that there exist incompatible experimental setups. For example, as it was originally initiated by Werner Heisenberg [1], the measurements of position and momentum of a quantum particle cannot be performed simultaneously unless some imprecisions are introduced [2]. The fact that position and momentum are not experimentally compatible physical quantities reflects the very properties of the quantum theory leading to the concept of coexistence.

In general, the coexistence of quantum devices means that they can be implemented as parts of a single device. Until now, the coexistence relation has been studied among quantum effects and observables; see, e.g., [3] and references therein. However, in addition to observed measurement outcome statistics, we can also end up with a quantum system. Quantum operations and instruments are used to mathematically describe both probabilities of the observed measurement outcomes and conditional states of the measured quantum system post-selected according to observed outcomes. Compared to an effect, an operation describes a particular result of a quantum measurement on a different level, providing more details about what happened during the measurement. The topic of this paper—the coexistence of quantum operations—is thus a natural extension of the previous studies of the coexistence of effects.

Let us now fix the notation and set the problem in mathematical terms. Let $\mathcal{H}$ be a complex separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ and $\mathcal{T}(\mathcal{H})$ the Banach spaces of bounded operators and trace class operators on $\mathcal{H}$, respectively. The set of quantum states (i.e. positive trace one operators) is denoted by $\mathcal{S}(\mathcal{H})$, and the set of quantum effects (i.e. positive operators bounded by the identity) is denoted by $\mathcal{E}(\mathcal{H})$.

An *operation* $\Phi$ is a completely positive linear mapping on $\mathcal{T}(\mathcal{H})$ such that

$$0 \leqslant \mathrm{tr}[\Phi(\varrho)] \leqslant 1,$$

for every $\varrho \in \mathcal{S}(\mathcal{H})$. An operation represents a probabilistic state transformation. Namely, if $\Phi$ is applied on an input state $\varrho$, then the state transformation $\varrho \mapsto \Phi(\varrho)$ occurs with the probability $\mathrm{tr}[\Phi(\varrho)]$, in which case the output state is $\Phi(\varrho)/\mathrm{tr}[\Phi(\varrho)]$. A special class is formed by operations satisfying $\mathrm{tr}[\Phi(\varrho)] = 1$ for every state $\varrho \in \mathcal{S}(\mathcal{H})$; these are called *channels,* and they describe deterministic state transformations.

An *instrument* is a device which takes as an input a quantum state, produces a measurement outcome and conditional to the measurement outcome also produces an output state. Mathematically, the instrument is represented as an operation-valued measure [4]. To be more precise, let $\mathcal{O}$ be a set of measurement outcomes and $\mathcal{F}$ a $\sigma$-algebra of subsets of $\mathcal{O}$. An instrument $\mathcal{J}$ is a $\sigma$-additive mapping $X \mapsto \mathcal{J}(X) \equiv \mathcal{J}_X$ from $\mathcal{F}$ to the set of operations on $\mathcal{T}(\mathcal{H})$. It is required to satisfy the normalization condition

$$\mathrm{tr}[\mathcal{J}_{\mathcal{O}}(\varrho)] = 1 \qquad \forall \varrho \in \mathcal{S}(\mathcal{H}),$$

which means that some state transformation occurs with probability 1. We denote by ran $\mathcal{J}$ the range of $\mathcal{J}$, that is,

$$\mathrm{ran}\, \mathcal{J} = \{\mathcal{J}_X \mid X \in \mathcal{F}\}.$$

**Definition 1.** *Two operations $\Phi$ and $\Psi$ are coexistent if there exists an instrument $\mathcal{J}$ such that $\Phi, \Psi \in \mathrm{ran}\, \mathcal{J}$.*

The mathematical coexistence problem we study in this paper is to find out whether two given operations are coexistent or not.

This definition is analogous to the definition of effect coexistence. We recall that two effects are coexistent if there exists an observable (POVM), which has both these effects in its range [5]. Since operations are more complex objects than effects, one naturally expects the coexistence relation for operations to be more complicated. On the other hand, the coexistence of effects in general is still an open problem, and it has turned out unexpectedly difficult to solve the effects coexistence problem even for quite restricted classes of effects. It is, therefore, not to be expected to find a complete solution of the operation coexistence problem here. Our intention is to formulate the problem properly and clarify the complexity of the tasks involved. As examples, we analyze the coexistence properties of Lüders operations and conditional state preparators.

The rest of this paper is organized as follows. In section 2, we derive two alternative mathematical formulations for the coexistence relation. In section 3, we study a special case of coexistence, called trivial coexistence. We show that pure (i.e. rank-1) operations can be coexistent only under very restrictive conditions. In section 4, we compare the coexistence of operations to the coexistence of the associated effects. Finally, section 5 gives our conclusions and an outlook.

## 2. General coexistence criteria for operations

We first shortly recall some basic concepts related to quantum operations as these are needed in the formulation of coexistence criteria. For more details on operations and instruments, we refer the reader to [4].

Let $\Phi$ be an operation. The dual mapping $\Phi^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ of $\Phi$ is defined by the duality formula,

$$\mathrm{tr}[\Phi^*(R)T] = \mathrm{tr}[R\Phi(T)], \tag{1}$$

required to hold for all $R \in \mathcal{L}(\mathcal{H}), T \in \mathcal{T}(\mathcal{H})$. The dual operation $\Phi^*$ describes the same quantum operation as $\Phi$ but in the Heisenberg picture. Setting $R = I$ in equation (1) we see that $\Phi$ determines a unique effect $A$ by the formula

$$\Phi^*(I) = A. \tag{2}$$

We often use the subscript notation $\Phi_A$ to emphasize this connection, meaning both $A$ and $\Phi_A$ give rise to the same measurement outcome probabilities. As operations also describe state transformations, it is understandable that the relation $A \to \Phi_A$ is one to many rather than one to one.

Two useful alternative mathematical descriptions of operations are provided by *Kraus decomposition* [6] and *Choi–Jamiolkowski isomorphism* [7, 8]. The latter description we formulate only in the case of a finite-dimensional Hilbert space.

The Kraus decomposition theorem states that a linear mapping $\Phi$ is an operation if and only if there exists a countable set of bounded operators $\{X_k\}$ such that $\sum_k X_k^* X_k \leqslant I$ and

$$\Phi(\varrho) = \sum_k X_k \varrho X_k^* \tag{3}$$

holds for all $\varrho \in \mathcal{S}(\mathcal{H})$. Further we also use a short-hand notation of equation (3) in the form $\Phi = \sum_k X_k \cdot X_k^*$. Using a Kraus decomposition (3) for $\Phi$ we see that equation (2) is equivalent to the condition

$$A = \sum_k X_k^* X_k.$$

For a fixed operation $\Phi$, the choice of operators $X_k$, referred to as Kraus operators, is not unique. Namely, two sets $\{X_k\}$ and $\{Y_l\}$ determine the same operation if and only if there are complex numbers $U_{lk}$ such that $\sum_l \overline{U}_{lj} U_{lk} = \delta_{jk}$ and $Y_l = \sum_k U_{lk} X_k$ [9]. When comparing two Kraus decompositions we can always assume that they have the same number of elements by adding null operators $O$ if necessary. With this assumption, the numbers $U_{lk}$ form a unitary matrix $U$.

The essential ingredient of the Choi–Jamiolkowski isomorphism is the so-called maximally entangled state:

$$\psi_+ = \frac{1}{\sqrt{d}} \sum_j \varphi_j \otimes \varphi_j,$$

where the vectors $\{\varphi_j\}$ form an orthonormal basis of $\mathcal{H}$ and $d = \dim \mathcal{H} < \infty$. We denote by $\mathcal{I}$ the identity operation $\mathcal{I}(\varrho) = \varrho$. The formulae,

$$\Phi \mapsto \Xi_\Phi = (\Phi \otimes \mathcal{I})(|\psi_+\rangle\langle\psi_+|), \tag{4}$$

$$\Xi \mapsto \Phi_\Xi, \quad \Phi_\Xi(\varrho) = d\,\mathrm{tr}_1[(\varrho^T \otimes I)\Xi], \tag{5}$$

determine a one-to-one Choi–Jamiolkowski mapping (and its inverse) between linear mappings $\Phi$ on $\mathcal{T}(\mathcal{H})$ and operators $\Xi$ on $\mathcal{H} \otimes \mathcal{H}$. The transposition is with respect to the orthonormal basis $\{\varphi_j\}$ used in the definition of the maximally entangled state $\psi_+$.

If $\Phi$ is an operation, then $\Xi_\Phi$ is positive and

$$d\,\mathrm{tr}_1[\Xi_\Phi] = \left(\sum_k X_k^* X_k\right)^T \leqslant I.$$

The reverse of this statement is also true; hence, any positive operator $\Xi$ on $\mathcal{H}\otimes\mathcal{H}$ induces an operation provided that $d\,\mathrm{tr}_1[\Xi] \leqslant I$. Thus, under the Choi–Jamiolkowski isomorphism, operations on $\mathcal{H}$ are associated with a specific subset of operators on $\mathcal{H}\otimes\mathcal{H}$. Unlike Kraus operators, the Choi–Jamiolkowski operator $\Xi_\Phi$ for a given operation $\Phi$ is unique. In terms of Choi–Jamiolkowski operators, equation (2) reads

$$d\,\mathrm{tr}_1[(\Phi\otimes\mathcal{I})(|\psi_+\rangle\langle\psi_+|)] = A^T. \tag{6}$$

For each operation $\Phi$, there is a minimal number of operators $X_k$ needed in its Kraus decomposition. This number is called the (*Kraus*) *rank* of $\Phi$, and we call any Kraus decomposition with this minimal number of elements a *minimal Kraus decomposition*. (Note, however, that even the choice of minimal Kraus decomposition is not unique.) Moreover, the rank of the associated Choi–Jamiolkowski operator $\Xi_\Phi$ is equal to the Kraus rank of the operation $\Phi$, i.e. $\mathrm{rank}(\Phi) \equiv \mathrm{rank}(\Xi_\Phi)$. Operations with Kraus rank 1 are called *pure*. They are exactly the extremal elements in the convex set of all operations.

The following two classes of operations, namely, conditional state preparators and Lüders operations, will be used later to exemplify the coexistence conditions.

**Example 1.** A *conditional state preparator* is an operation $\Phi_A^\xi$ of the form

$$\Phi_A^\xi(\varrho) = \mathrm{tr}[\varrho A]\xi$$

for some fixed $\xi \in \mathcal{S}(\mathcal{H})$ and $A \in \mathcal{E}(\mathcal{H})$. If $A = I$, then this operation is just the constant mapping, $\varrho \mapsto \xi$. For the conditional state preparator $\Phi_A^\xi$, the associated Choi–Jamiolkowski operator $\Xi_A^\xi$ reads

$$\Xi_A^\xi = \frac{1}{d}\sum_{j,k}\Phi_A^\xi(|\varphi_j\rangle\langle\varphi_k|)\otimes|\varphi_j\rangle\langle\varphi_k| = \frac{1}{d}\xi\otimes A^T.$$

**Example 2.** Let $A$ be an effect. The *Lüders operation* $\Phi_A^{\mathcal{L}}$ associated with $A$ is defined by the formula

$$\Phi_A^{\mathcal{L}}(\varrho) = \sqrt{A}\varrho\sqrt{A}.$$

The associated Choi–Jamiolkowski operator $\Xi_A^{\mathcal{L}}$ is given by

$$\Xi_A^{\mathcal{L}} = |(\sqrt{A}\otimes I)\psi_+\rangle\langle(\sqrt{A}\otimes I)\psi_+| \equiv |\psi_A\rangle\langle\psi_A|, \tag{7}$$

with

$$\mathrm{tr}\big[\Xi_A^{\mathcal{L}}\big] = \langle\psi_A|\psi_A\rangle = \frac{1}{d}\,\mathrm{tr}[A] \leqslant 1.$$

If $A = I$, then the corresponding Lüders operation is the identity operation $\mathcal{I}$, and we get $\Xi_I^{\mathcal{L}} = |\psi_+\rangle\langle\psi_+|$.

Let us also note that if $A$ is proportional to a one-dimensional projection $P$, i.e., $A = \lambda P$ for some $0 \leqslant \lambda \leqslant 1$, then $\Phi_A^{\mathcal{L}}$ is the conditional state preparator $\Phi_A^P$. Namely, we have

$$\sqrt{A}\varrho\sqrt{A} = \lambda P\varrho P = \mathrm{tr}[\varrho\lambda P]P = \mathrm{tr}[\varrho A]P.$$

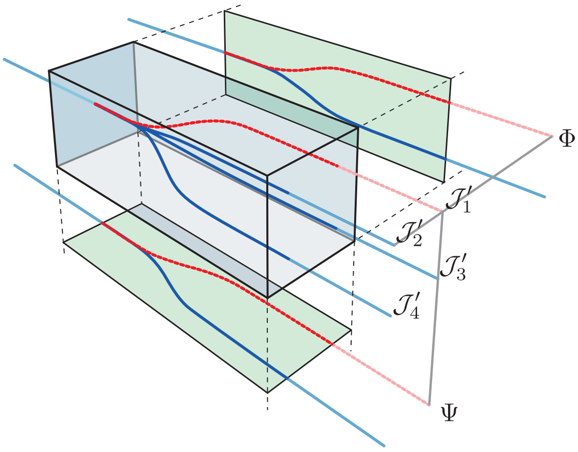


**Figure 1.** Coexistence of two operations $\Psi$ and $\Phi$ is conditioned by the existence of a four-outcome instrument determined by four operations $\mathcal{J}'_1$ through $\mathcal{J}'_4$ such that $\mathcal{J}'_1 + \mathcal{J}'_2 = \Phi$ and $\mathcal{J}'_1 + \mathcal{J}'_3 = \Psi$. (This figure is in colour only in the electronic version)

In all other cases, when rank $A > 1$, Lüders operation $\Phi_A^{\mathcal{L}}$ is not a conditional state preparator since for a conditional state preparator $\Phi_A^{\xi}$ we have $\mathrm{rank}\left(\Xi_A^{\xi}\right) = \mathrm{rank}(\xi)\mathrm{rank}(A^T) \geqslant \mathrm{rank}(A) > 1$, but $\mathrm{rank}\left(\Xi_A^{\mathcal{L}}\right) = 1$.

We now make a simple but useful observation related to definition 1. Suppose that $\Phi$ and $\Psi$ are coexistent operations and that $\mathcal{J}$ is an instrument such that $\Phi, \Psi \in \mathrm{ran}\,\mathcal{J}$. This means that there are outcome sets $X, Y$ such that $\mathcal{J}_X = \Phi$, $\mathcal{J}_Y = \Psi$. We define another instrument $\mathcal{J}'$ with outcomes $\{1, 2, 3, 4\}$ by setting

$$\mathcal{J}'_1 = \mathcal{J}_{X\cap Y}, \qquad \mathcal{J}'_2 = \mathcal{J}_{X\cap\neg Y}, \qquad \mathcal{J}'_3 = \mathcal{J}_{\neg X\cap Y}, \qquad \mathcal{J}'_4 = \mathcal{J}_{\neg X\cap\neg Y}. \tag{8}$$

It follows from the properties of $\mathcal{J}$ that $\mathcal{J}'$ is indeed an instrument. The operations $\Phi$ and $\Psi$ are in the range of $\mathcal{J}'$ as $\mathcal{J}'_1 + \mathcal{J}'_2 = \Phi$ and $\mathcal{J}'_1 + \mathcal{J}'_3 = \Psi$. Thus, we conclude the following.

**Proposition 1.** *Two operations are coexistent if and only if they are in the range of an instrument defined on the outcome set* $\mathcal{O} = \{1, 2, 3, 4\}$.

The fact stated in proposition 1 simplifies the study of the coexistence relation as we need to concentrate only on four outcome instruments. An illustration of a four-outcome instrument and two coexistent operations is depicted in figure 1.

**Proposition 2.** *Two operations $\Phi$ and $\Psi$ are coexistent if and only if there exists a sequence of bounded operators $\{X_j\}_{j\in J}$ and index subsets $J_1, J_2 \subseteq J$ such that*

$$\Phi(\cdot) = \sum_{j\in J_1} X_j \cdot X_j^*, \qquad \Psi(\cdot) = \sum_{j\in J_2} X_j \cdot X_j^* \tag{9}$$

*and*

$$\sum_{j\in J} X_j^* X_j = I. \tag{10}$$

*If $\Phi$ and $\Psi$ are coexistent, we can choose an index set $J$ with at most $3d^2 + 1$ elements.*

**Proof.** Suppose first that there exists a sequence of bounded operators $\{X_j\}_{j\in J}$ with the required properties. By defining $\mathcal{J}_j(\rho) = X_j\rho X_j^*$, we get an instrument $\mathcal{J}$ having both $\Phi$ and $\Psi$ in its range. Hence, $\Phi$ and $\Psi$ are coexistent.

Suppose then that $\Phi$ and $\Psi$ are coexistent. As we have seen in proposition 1, there exists a four-outcome instrument $\mathcal{J}$ such that $\mathcal{J}_1 + \mathcal{J}_2 = \Phi$ and $\mathcal{J}_1 + \mathcal{J}_3 = \Psi$. Choose a Kraus decomposition $\{X_j^{(k)}\}$ for each $\mathcal{J}_k$. The union $\cup_k\{X_j^{(k)}\}$ forms a collection with the required properties.

The last claim follows by noting that each operation $\mathcal{J}_k, k = 1, 2, 3$, has a Kraus decomposition with (at most) $d^2$ Kraus operators. On the other hand, the role of the operation $\mathcal{J}_4$ is only to guarantee the normalization of the instrument $\mathcal{J}$. We can hence re-define $\mathcal{J}_4$ as the operation having a single Kraus operator $\sqrt{I - \sum_{k=1}^3 \mathcal{J}_k^*(I)}$. □

Let us note that the statement of proposition 2 remains valid if equation (10) is replaced with an inequality,

$$\sum_{j\in J} X_j^* X_j \leqslant I, \tag{11}$$

and then the number of elements in $J$ can be chosen to be at most $3d^2$. We can hence use either condition (10) or (11), depending on which one happens to be more convenient.

In the following, we formulate the basic coexistence criterion of proposition 2 in terms of Choi–Jamiolkowski operators.

**Proposition 3.** *Two operations $\Phi$ and $\Psi$ are coexistent if and only if there exists a state $\Omega \in \mathcal{S}(\mathcal{H}\otimes\mathcal{H})$ with $d\,\mathrm{tr}_1[\Omega] = I$, which has a decomposition into four positive operators $\Omega = \sum_{k=1}^4 \Xi_k$ such that*

$$\Xi_\Phi = \Xi_1 + \Xi_2, \qquad \Xi_\Psi = \Xi_1 + \Xi_3. \tag{12}$$

**Proof.** In the Choi–Jamiolkowski representation, a four-outcome instrument $\mathcal{J}$ translates into a mapping $k \mapsto \Xi_k = (\mathcal{J}_k \otimes \mathcal{I})(|\psi_+\rangle\langle\psi_+|)$, where $\Xi_k$ are positive operators on $\mathcal{H}\otimes\mathcal{H}$ and $\sum_{k=1}^4 d\,\mathrm{tr}_1[\Xi_k] = I$ meaning that $\Omega \equiv \sum_{k=1}^4 \Xi_k$ is a state in $\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. The claim then follows from proposition 1. □

**Example 3.** Let $U$ be a unitary operator and $\mathcal{U}$ the corresponding unitary channel, i.e., $\mathcal{U}(\varrho) = U\varrho U^*$. As $\mathcal{U}$ describes a deterministic and reversible state transformation, it is not expected to be coexistent with many other operations. Of course, we can reduce $\mathcal{U}$ by accepting the state transformation with some probability $0 \leqslant \lambda \leqslant 1$ and ignoring the rest, hence obtaining an operation $\lambda\mathcal{U}$. Thus, $\mathcal{U}$ and $\lambda\mathcal{U}$ are coexistent operations.

A proof that $\lambda\mathcal{U}$ are indeed the only operations coexistent with the unitary channel $\mathcal{U}$ can be seen from proposition 3. The Choi–Jamiolkowski operator $\Xi_\mathcal{U}$ corresponding to $\mathcal{U}$ is $|\psi_U\rangle\langle\psi_U|$, where

$$\psi_U = \frac{1}{\sqrt{d}}\sum_j U\varphi_j \otimes \varphi_j.$$

In particular, $\Xi_\mathcal{U}$ is a one-dimensional projection, and it can be written as a sum of two positive operators only if they are proportional to $\Xi_\mathcal{U}$. On the other hand, there cannot be other operators in the decomposition of $\Omega$ as $\Xi_\mathcal{U}$ is already normalized, $d\,\mathrm{tr}_1[\Xi_\mathcal{U}] = I$. Therefore, an operation $\Phi$ is coexistent with $\mathcal{U}$ only if $\Xi_\Phi = \lambda\Xi_\mathcal{U}$ for some number $0 \leqslant \lambda \leqslant 1$, which means that $\Phi = \lambda\mathcal{U}$.

## 3. Trivial coexistence

Let $\Phi$ and $\Psi$ be two coexistent operations. Referring to proposition 2 we can meet with three possible situations:

(C1) $J_1 \cap J_2 = \emptyset$,

(C2) $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$,

(C3) none of the above.

In particular, if we can choose index subsets $J_1$, $J_2$ such that $J_1 \cap J_2 = \emptyset$, then $\Psi + \Phi$ is also an operation. Similarly, (C2) implies that $\Psi - \Phi$ or $\Phi - \Psi$ is an operation. It is clear that the verification of the coexistence of $\Psi$ and $\Phi$ in such cases is straightforward.

We conclude that the coexistence of $\Phi$ and $\Psi$ falls out trivially if

(T1) $\Phi + \Psi$ is an operation,

(T2) $\Phi - \Psi$ or $\Psi - \Phi$ is an operation;

thus, two operations satisfying one of the conditions (T1), (T2) are called *trivially coexistent*.

For a general coexistence problem, we need to consider four outcome instruments; however, in the case of trivial coexistence three outcome instruments are sufficient. For instance, if $\Psi + \Phi$ is an operation, then we can choose a channel $\mathcal{E}$ such that $\mathcal{E} - \Psi - \Phi$ is also an operation and we can define a three-outcome instrument $\mathcal{J}_1 = \Phi$, $\mathcal{J}_2 = \Psi$, $\mathcal{J}_3 = \mathcal{E} - \Phi - \Psi$. Similarly, if $\Psi - \Phi$ is an operation, we choose a channel $\mathcal{E}$ such that $\mathcal{E} - \Psi$ is an operation, and we can define a three-outcome instrument $\mathcal{J}_1 = \Phi$, $\mathcal{J}_2 = \Psi - \Phi$, $\mathcal{J}_3 = \mathcal{E} - \Psi$.

Let $\Phi_A$ and $\Psi_B$ be two operations. The condition (T1) means

$$\mathrm{tr}[\Phi_A(\varrho) + \Psi_B(\varrho)] \leqslant 1,$$

for every $\varrho \in \mathcal{S}(\mathcal{H})$. This is equivalent to

$$A + B \leqslant I.$$

The set of operations is a partially ordered set if we adopt the following relation between operations:

$$\Psi \leqslant \Phi \quad \Leftrightarrow \quad \Phi - \Psi \text{ is an operation.}$$

Hence, we can write (T2) as

$$\Psi \leqslant \Phi \qquad \text{or} \qquad \Phi \leqslant \Psi.$$

Unlike (T1), these conditions do not reduce to generally valid effect inequalities in terms of the associated effects $A$ and $B$. Let us note that a necessary condition for $\Phi_A \leqslant \Psi_B$ is that $A \leqslant B$. This is, however, not generally sufficient. The form of (T2) depends on the operations in question, as we demonstrate in examples 4 and 5 below.

**Example 4.** Let $\xi$ be a fixed state, $A$ and $B$ two effects, and $\Phi_A^\xi$ and $\Phi_B^\xi$ the corresponding conditional state preparators. The trivial coexistence condition (T2) now becomes a requirement that either $A \geqslant B$ or $B \geqslant A$.

**Proposition 4.** *Let $\Phi$ and $\Psi$ be two pure operations. If $\Phi$ and $\Psi$ are coexistent, then they are trivially coexistent.*

**Proof.** As $\Phi$ and $\Psi$ are pure, there are bounded operators $V$, $W$ such that $\Phi(\cdot) = V \cdot V^*$ and $\Psi(\cdot) = W \cdot W^*$. By proposition 2 the coexistence of $\Phi$ and $\Psi$ means that

$$\Phi(\cdot) = \sum_{j \in J_1} X_j \cdot X_j^* = V \cdot V^*, \qquad \Psi(\cdot) = \sum_{j \in J_2} X_j \cdot X_j^* = W \cdot W^*.$$

It follows that $X_j = c_j V$ for every $j \in J_1$ and $X_j = d_j W$ for every $j \in J_2$. Here $c_j, d_j$ are non-zero complex numbers. This shows that we have two possibilities: either $J_1 \cap J_2 = \emptyset$ or $c_j V = d_j W$ for some $j \in J_1 \cap J_2$. The first case means that $\Phi + \Psi$ is an operation, while the second case leads to the condition $\Phi = p\Psi$ for some $p \in \mathbb{R}_+$. This implies that either $\Phi - \Psi$ or $\Psi - \Phi$ is an operation, depending on whether $p \leqslant 1$ or $p \geqslant 1$. □

**Example 5.** Lüders operations are, by definition, pure operations, and therefore their coexistence reduces to the trivial coexistence by proposition 4. Different to example 4, the effect inequalities $A \leqslant B$, $B \leqslant A$ are not sufficient to guarantee the coexistence of Lüders operations $\Phi_A^{\mathcal{L}}$ and $\Phi_B^{\mathcal{L}}$. For example, let $P$ be a one-dimensional projection. Then $P \leqslant I$, but neither $\Phi_P^{\mathcal{L}} - \Phi_I^{\mathcal{L}}$, nor $\Phi_I^{\mathcal{L}} - \Phi_P^{\mathcal{L}}$ nor $\Phi_I^{\mathcal{L}} + \Phi_P^{\mathcal{L}}$ are operations. By proposition 4 we thus conclude that $\Phi_P^{\mathcal{L}}$ and $\Phi_I^{\mathcal{L}}$ are not coexistent operations.

To see the content of the condition (T2), suppose that $\Phi_A^{\mathcal{L}} - \Phi_B^{\mathcal{L}}$ is an operation. This implies that

$$|\sqrt{A}\psi\rangle\langle\sqrt{A}\psi| \geqslant |\sqrt{B}\psi\rangle\langle\sqrt{B}\psi|,$$

for every $\psi \in \mathcal{H}$. As a consequence, $B = \lambda A$ for some $0 \leqslant \lambda \leqslant 1$.

In summary, Lüders operations are coexistent if and only if either $A$ is proportional to $B$ or $A + B \leqslant I$. The effects $A$ and $B$, for which the latter inequality holds, are called *disjoint*.

## 4. Coexistence of operations versus coexistence of effects

As we have seen in section 3, the trivial coexistence of operations is closely related to the coexistence of effects. In this section, we investigate in more details the relations between the coexistences of operations, their associated effects and Choi–Jamiolkowski operators.

**Proposition 5.** *If two operations $\Phi_A$ and $\Psi_B$ are coexistent, then the corresponding effects $A$ and $B$ are coexistent.*

**Proof.** By proposition 2, the coexistence of $\Phi_A$ and $\Psi_B$ is equivalent to the existence of a set $\{X_j\}_{j\in J}$ and index subsets $J_1, J_2 \subseteq J$ such that

$$\Phi_A(\cdot) = \sum_{j\in J_1} X_j \cdot X_j^*, \qquad \Psi_B(\cdot) = \sum_{j\in J_2} X_j \cdot X_j^*.$$

For each $j \in J$, we define $\mathsf{G}_j := X_j^* X_j$. The effects $\mathsf{G}_j$ define a discrete observable $\mathsf{G}$. The effects $A$ and $B$ belong to the range of $\mathsf{G}$ as $A = \sum_{j\in J_1} X_j^* X_j$ and $B = \sum_{j\in J_2} X_j^* X_j$. Therefore, $A$ and $B$ are coexistent. □

**Example 6.** Let $A$ and $B$ be two coexistent effects. This means that there is an observable $\mathsf{G}$ such that $\mathsf{G}(X) = A$ and $\mathsf{G}(Y) = B$. We fix a state $\xi$ and define an instrument $\mathcal{J}$ by the formula

$$\mathcal{J}_Z(\varrho) = \mathrm{tr}[\varrho \mathsf{G}(Z)]\xi.$$

The operations $\mathcal{J}_X$ and $\mathcal{J}_Y$ are then the conditional state preparators $\Phi_A^{\xi}$ and $\Phi_B^{\xi}$, respectively. Thus, if $A$ and $B$ are coexistent, then the conditional state preparations $\Phi_A^{\xi}$ and $\Phi_B^{\xi}$ are also coexistent.

As discussed in section 2, each operation $\Phi$ determines a Choi–Jamiolkowski operator $\Xi_\Phi$ on $\mathcal{H}\otimes\mathcal{H}$ such that $d\,\mathrm{tr}_1[\Xi_\Phi] \leqslant I$. As Choi–Jamiolkowski operators are effects on $\mathcal{H}\otimes\mathcal{H}$, we can formally consider their coexistence. Our aim is to investigate the relation between the coexistence of operations and the coexistence of Choi–Jamiolkowski operators as effects.

**Table 1.** Relations between coexistence of effects and coexistence of their compatible operations.

| | | |
|---|---|---|
| $A$ and $B$ are not coexistent | $\Longrightarrow$ | There are no coexistent operations $\Phi_A$ and $\Psi_B$ |
| $A$ and $B$ are coexistent | $\Longrightarrow$ | There exist coexistent operations $\Phi_A$ and $\Psi_B$ |
| $A + B \leqslant I$ | $\Longrightarrow$ | All operations $\Phi_A$ and $\Psi_B$ are trivially coexistent |
| $\Xi_\Phi$ and $\Xi_\Psi$ are not coexistent | $\Longrightarrow$ | $\Phi$ and $\Psi$ are not coexistent |

If $\Phi$ and $\Psi$ are coexistent, then the linearity of the Choi–Jamiolkowski isomorphism guarantees that effects $\Xi_\Psi$ and $\Xi_\Phi$ are coexistent, too. However, the converse is not true. Namely, even if $\Xi_\Phi$ and $\Xi_\Psi$ are coexistent as effects, the associated operations $\Phi, \Psi$ need not be coexistent. For example, according to proposition 4, two rank-1 operations $\Phi_A$ and $\Psi_B$ are coexistent only if they are trivially coexistent. However, if $A, B$ are one-dimensional projections associated with vectors $\varphi, \eta \in \mathcal{H}$, then $\Xi_A = \frac{1}{d}|\varphi\rangle\langle\varphi| \otimes (|\varphi\rangle\langle\varphi|)^T$ and $\Xi_B = \frac{1}{d}|\eta\rangle\langle\eta| \otimes (|\eta\rangle\langle\eta|)^T$, which are always (trivially) coexistent as effects, because $\Xi_A + \Xi_B \leqslant I \otimes I$. The point is that $I \otimes I$ does not correspond to any operation, because $d\mathrm{tr}_1[I \otimes I] = d^2 I \nleqslant I$.

Table 1 summarizes the mentioned results. We see that the remaining problem is the following: if $A$ and $B$ are coexistent effects but do not satisfy $A + B \leqslant I$, then what are the coexistent operations $\Phi_A$ and $\Psi_B$? The following examples demonstrate different aspects of this general problem.

**Example 7.** Let $\Phi_A^{\xi_1}$ and $\Phi_B^{\xi_2}$ be two conditional state preparators such that $\xi_1$ and $\xi_2$ are pure states, i.e., $\xi_i = |\phi_i\rangle\langle\phi_i|$ for some unit vectors $\phi_i \in \mathcal{H}$. A Kraus decomposition of $\Phi_A^{\xi_1}$, with Kraus operators $X_k$, is of the form $|\phi_1\rangle\langle\eta_k|$ for some vector $\eta_k$, or a sum of these kinds of operators. Similarly, a Kraus decomposition of $\Phi_B^{\xi_2}$, with Kraus operators $Y_l$, is either $|\phi_2\rangle\langle\eta_l'|$ for some vector $\eta_l'$, or a sum of these kinds of operators. Suppose that $\Phi_A^{\xi_1}$ and $\Phi_B^{\xi_2}$ are coexistent, but the inequality $A + B \leqslant I$ does not hold. This implies that $X_k = Y_l$ for some indices $k$ and $l$. As a consequence, we must have $\xi_1 = \xi_2$. We conclude that if $\xi_1 \neq \xi_2$ then the conditional pure state preparators are coexistent only if $A + B \leqslant I$.

**Example 8.** It is customary to call an effect $A$ *trivial* if it is of the form $\lambda I$ for some $0 \leqslant \lambda \leqslant 1$. Trivial effects are exactly those effects which are coexistent with all the other effects.

In the same way, we can call an operation trivial if it is coexistent with all the other operations. Clearly, the null operation $\Phi_O^{\mathcal{C}}(\varrho) = O$ is trivial in this sense, since any instrument can be expanded by adding one additional outcome and attaching $\Phi_O^{\mathcal{C}}$ to this additional outcome.

Actually, the null operation is the only trivial operation. As shown in example 3 a unitary channel $\mathcal{U}$ is coexistent only with operations $\lambda\mathcal{U}$. Since a trivial operation is coexistent with all unitary channels, it must be the null operation.

## 5. Discussion

In this paper, we have studied the coexistence of two quantum operations. In particular, we have shown that two common types of operations in quantum information, namely conditional state preparations and Lüders operations, are coexistent only under some very restrictive conditions. We have also shown that the coexistence problem for operations does not reduce to the coexistence problem for effects.

Recently, coexistence of two arbitrary qubit effects has been characterized [3, 10–12]. It would be interesting to give an analogous characterization of two arbitrary qubit operations.

This problem, however, seems to be much more intricate as already the parametrization of the qubit operations is quite a complex task [13].

In quantum information theory, it has become typical to consider impossible devices, forbidden by the rules of quantum mechanics. For an impossible device, one can then study its best approximate substitute. Especially, we can ask for the best coexistent approximations for two non-coexistent Lüders operations. This problem will be studied elsewhere.

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